Solutions to Maths workbook - 2 | Permutation & Combination

Level – 3	Daily Tutorial Sheet – 12

216. Suppose each player scored less points against his compatriots than against players from other country. Let the players of first country be $A_1, A_2 \ldots A_n$ and second country. $B_1, B_2 \ldots B_n$.

Total matches = Total points $= {}^{2n}C_2 = n(2n-1)$.

Since, each player plays $(2n-1)$ matches, the range of points for a player is $\{0, 0.5, 1, 1.5 \ldots 2n-1\cdot 5, 2n-1\}$ i.e., 2n integral points and $(2n-1)$ non-integral points and with each of the 2n players having scored different points, by pigeon hole principle, at least 1 player must score integral points.

For $i \in \{1, 2, \ldots n\}$, let the points scored by A_i against his compatriots = k_i and against the rest = l_i.

Similarly, let the points scored by B_i against his compatriots be p_i and the rest be q_i

By our assumption $\ell_i > k_i$ for $i \in \{1, 2, \ldots n\}$

$\Rightarrow \quad \ell_i \geq k_i + 0.5$

$\Rightarrow \quad \sum_{i=1}^{n} \ell_i \geq \sum_{i=1}^{n} k_i + n/2$

Note that $= \sum_{i=1}^{n} k_i = {}^nC_2$ because $A_1, A_2 \ldots A_n$ play ${}^nC_2$ matches among each other.

$\Rightarrow \quad \sum_{i=1}^{n} \ell_i \geq n^2/2$

By the same argument, $\sum_{i=1}^{n} q_i \geq n^2/2$

Now, total n^2 matches involve a player of both countries $\quad \Rightarrow \quad \sum_{i=1}^{n} \ell_i + \sum_{i=1}^{n} q_i = n^2$

Which mean both the above inequalities are strict equalities i.e., $\sum_{i=1}^{n} \ell_i = \sum_{i=1}^{n} q_i = n^2/2$

$\Rightarrow \quad$ For $i \in \{1, 2, \ldots n\} \qquad \ell_i = k_i + 0.5$ and $q_i = p_i + 0.5$

$\Rightarrow \quad$ Points of $A_i = k_i + \ell_i$ for $i \in \{1, ..n\} = 2k_i + 0.5$

Similarly point of $B_i = 2pi + 0.5$ for $i \in \{1, \ldots n\}$

Thus, all the players have scored non-integral points which is a contradiction to the condition that all players have different points.

Hence, our assumption has been contradicted.

Thus, there is at least one player who scored at least as many points against his compatriots as against other players.

217. Take any magic square of line sum r and side length 3. It is clear that the four elements shown in the figure determine all the rest of the square.

a	d	
	b	
		c

Indeed, the next table shows our only possible choice for each remaining entry. Thus, all we need to do is to compute the number of ways we can choose a, b, c and d so that we indeed have that one choice, i.e., the obtained entries of the magic square are all non-negative.

a	d	$r-a-d$
$r+c-(a+d+b)$	b	$a+d-c$
$b+d-c$	$r-b-d$	c

The previous table shows that the entries of our matrix will be non-negative if and only if the following inequalities hold:

$a+d \le r$ (3.8)

$b+d \le r$ (3.9)

$c \le a+d$ (3.10)

$c \le b+d$ (3.11)

$a+d+b-c \le r$ (3.12)

We will consider three different cases, according to the position of the smallest element on the main diagonal. In each of them, at least three of the five conditions above will become redundant, and will only need to deal with the remaining one or two.

(a) Suppose $0 \le a \le b$ and $0 \le a \le c$. In this case conditions (3.8), (3.11), and (3.12) are clearly redundant, because they are implied by (3.9) and (3.10).

The crucial observation is that in all the three cases we can collect all our conditions into one single chain of inequalities. In this case we do it as follows:

$a \le c \le a+d \le b+d \le r$. (3.13)

Moreover, note that once we know the terms of this chain, then we know a, b, c and d, too, thus we have determined the magic square. Thus, all we need to do is simply count how many ways there are to choose these four terms. Inequality (3.13) shows that these terms are nondecreasing, therefore the number of ways there are to choose these four terms. Inequality (3.13) shows that these terms are nondecreasing, therefore the number of ways to choose them is simply the number of 4-combinations of r + 1 elements with repetitions allowed, which is $\binom{r+4}{4}$. (Recall that 0 is allowed to be an entry).

(b) Now suppose $a > b$ and $c \ge b$. Then (3.9), (3.11) and (3.12) are redundant. Consider the chain of inequalities $b \le c \le b+d \le a+d-1 \le r-1$. (3.14)

We can use the argument of the previous case as the roles of a and b are completely symmetric. The only change is that here we do not count those magic squares in which a = b, and this explains the (–1) in the last two terms. Thus, here we have to choose four elements in non-decreasing order out of the set {0, 1, ..., $r-1$}, which can be done in $\binom{r+3}{4}$ ways.

(c) Finally, suppose that $a > c$ and $b > c$. Then (3.8), (3.9), (3.10) and (3.11) are redundant. Condition (3.12) and our assumptions can be collected into the following chain:

$c \le b-1 \le b+d-1 \le a+b+d-c-2 \le r-2$ (3.15)

Here the first inequality is equivalent to our assumptions $c < b$, the second one says that d is non-negative, the third one is equivalent to our assumption $c \le a-1$, and the last one is equivalent to (3.12). The four terms of (3.15) determine a, b, c and d, and they can be chosen in $\binom{r+2}{4}$ ways, which completes the proof.

Thus, the number of 3×3 magic squares of line sum r is indeed $\binom{r+4}{4}+\binom{r+3}{4}+\binom{r+2}{4}$.

Furthermore, the three terms in this sum count the magic squares in which the (first) minimum element of the main diagonal is the first, second, or third element.

218. This follows directly from the multinomial theorem by substituting $x_1 = x_2 = x_3 = 1$.

219. The set A = {1, 2, 3, ..., 100} contains 100 elements. There are 9 elements of form r mod(11) where r = 0, 2, 3, ...11 while there are 10 elements of the form 1 mod(11).

Case-I: If among the chosen 53 elements, two of them are of the form 0 mod(11), their sum will be obviously divisible by 11.

Case-II: If among the chosen 53 elements the number of elements lying in 0 mode (11), is less than or equal to 1, then at least six of, r mod (11) where r = {1, 2, ... 10}, residue groups must be chosen using P.H.P, there exist $k, \ell \in r$ such that $k + \ell = 11$, therefore the proof.

In fact, even a subset of 48 elements will have two such elements.

220. There are 34 integers in the A.P. 1, 4, 7,...,100. Let us denote the set $\{1,4,7,...,100\}$ by S. Let us group then into 17 pairs $(4,100),(7,97),(10,94),...(49,55)$, and $(1,52)$. The sum of the integers in each of the first sixteen pairs is 104. The last pair consist of the two integers which cannot be paired with any other integer in the given A.P. so as to have the sum 104. Let us try to construct a subset of S which is as big as possible, and the property that no two numbers of the set add up to 104. Such a set can have at the most 18 members, namely the two integers 1 and 100, and exactly one out of each of the remaining 16 pairs.

The moment we add one more member of S to it, it will have both the members of one of the sixteen pairs (4,100) ... (49, 55), i.e., it will have two distinct integers whose sum is 104.

Therefore, in any set 20 (in fact 19!) distinct integers chosen from S, there will always be two distinct integers whose sum is 104.

221.(1) Let us call the friends *a*, *b*, *c*, *d*, *e* and f. Since I dined with all the six on exactly one day, therefore at exactly one dinner I had all the six friends.

Since I dined with every five of them on two days, I must have had six dinners in which I dined with *b, c, d, e, f; a, c, d, e, f; a, b, d, e, f; a, b, c, e, f; a, b, c, d, f* respectively.

The above seven (1+6) dinners are such that every four of the friends where there in exactly 3 of them, and every three of them where there in exactly 4 of them.

Two of the friends, say a and b, where present at the first dinner, and at four other dinners, that is at five dinners. Likewise, every other pair of the friends was present at five of the above dinners.

Each of the friends was present at the first dinner and five others, that is at six dinners.

Since each one of the friends was present at seven dinners in all, I must have had one dinner exclusively with each one of the six friends. It is clear that each one of the friends was absent at five of these six dinners, and therefore in all absent at six out of the thirteen dinners considered so far.
Since a friend was absent at seven dinners, therefore I must have had one dinner alone.